SET THEORY TRAINING EXERCISES

Problem 1.

a) State König's Theorem.

b) Using König's Theorem, prove $\forall \kappa \in \text{Card} : \kappa < \kappa^{\text{cof}\kappa}$ (cardinal exponentiation).

c) From b), infer that for all $\alpha \in \text{Ord} : 2^\omega \neq \aleph_{\alpha+\omega}$ (cardinal exponentiation, ordinal addition).

Problem 2. Recall the von Neumann hierarchy $V_\alpha, \alpha \in \text{Ord}$.

a) Show $\forall A \exists \alpha \in \text{Ord} : A \in V_\alpha$ without using $\bigcup_{\alpha \in \text{Ord}} V_\alpha = V$. State why your proof requires the Axiom of Foundation.

b) Show $\forall \alpha \in \text{Ord} : \alpha \notin V_\alpha$.

Problem 3. Decide which of the following statements are true and which are false. You do not need to prove your answers. "club" and "stationary" are always to be understood with respect to $\kappa \in \text{Card}$.

a) If $\text{cof}\kappa > \omega$, and C_α, $\alpha < \text{cof}\kappa$ is a sequence of cub sets, then $\bigcap_{\alpha < \text{cof}\kappa} C_\alpha$ is cub.

b) If $\text{cof}\kappa > \omega$, the intersection of a stationary set and a cub set is stationary.

c) If $\kappa > \omega$, the intersection of finitely many cub sets is always cub.

d) If $\text{cof}\kappa \leq \omega$, the intersection of a stationary set and a cub set is cub.

e) If $\kappa = \text{cof}\kappa$, $\kappa \neq \omega$ and $S \subseteq \kappa$, then: S is stationary if and only if for all regressive functions f on S there is a stationary $T \subseteq S$ with $\text{card}(f[T]) = 1$.

Problem 4. Show: If $\kappa \leq \lambda$ are infinite cardinals, then $\lambda^\kappa = \mathrm{card}([\lambda]^{\leq\kappa})$.

Problem 5.

a) State the Axiom of Choice in First Order Logic.

b) Using the Axiom of Choice, show that the countable union of countable sets is again countable. Mark in your proof where you use the Axiom of Choice.

Problem 6.

a) State the Axiom of Infinity (informally, but precisely) and define ω.

b) Show that ω is the smallest limit ordinal.

c) Show that ω is closed under ordinal addition.